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Note on Mr. Davison's Paper on the Straining of the Earth's Crust in Cooling

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taking into consideration the whole surface of the globe, *the process of mountain-making diminishes with the increase of the time, and so also does the rate of continental evolution.*

(25) It cannot be said that the contraction theory on the hypothesis of solidity is entirely free from objections. Two very obvious ones have already been alluded to in the course of this paper, namely (1) The small calculated depth of the unstrained surface, especially in early geological periods; and (2) The small proportion of folded rock to stretched rock directly produced by secular cooling. But I do not think that these objections are by any means fatal to the theory. Assuming the Earth to be practically solid, and to have been originally at a high temperature throughout, I believe it may be concluded that the peculiar distribution of strain in the Earth's crust resulting from its secular cooling has contributed to the permanence of ocean-basins, and has been the main cause of the growth of continents and the formation of mountain chains.

VIII. *Note on Mr.* DAVISON'S *Paper on the Straining of the Earth's Crust in Cooling.*

By G. H. DARWIN, *LL.D., F.R.S., Fellow of Trinity College, and Plumian Professor in the University of Cambridge.*

Received June 15,—Read June 16, 1887.

MR. DAVISON'S interesting paper was, he says, suggested by a letter of mine published in 'Nature' on February 6, 1879. In that letter it is pointed out that the stratum of the Earth where the rate of cooling is most rapid lies some miles below the Earth's surface. Commenting on this, I wrote :—

"The Rev. O. FISHER very justly remarks that the more rapid contraction of the internal than the external strata would cause a wrinkling of the surface, although he does not admit that this can be the sole cause of geological distortion. The fact that the region of maximum rate of cooling is so near to the surface recalls the interesting series of experiments recently made by M. FAVRE ('Nature,' vol. 19, p. 108), where all the phenomena of geological contortion were reproduced in a layer of clay placed on a stretched india-rubber membrane, which was afterwards allowed to contract. Does it not seem possible that Mr. FISHER may have under-estimated the contractibility of rock in cooling, and that this is the sole cause of geological contortion?"

Mr. DAVISON works out the suggestion, and gives precision to the general idea contained in the letter. He shows, however, that there is a layer of zero strain in the

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Earth's surface, and that this layer, instead of that of greatest cooling, must be taken to represent FAVRE's elastic membrane.

It appears that the mathematical discussion of the problem in his paper is unnecessarily laborious,* and he has not made various important deductions as to the integral results of distortion and as to the magnitude of the effects to be expected. I, therefore, offer the following note with the intention of rendering more complete an important chapter in the mechanics of geology.

When a spherical shell expands with rise of temperature, it may be said to stretch, in one sense of the word. If the shell were one of the layers of the Earth, such a stretching would have no geological effect, for it would merely involve a change of density. The term "stretching" then requires an explanation in connection with Mr. DAVISON's paper. The stretching which we have to consider is, in fact, simply the excess of the actual stretching above that due to rise of temperature. The negative of such stretching is a contraction, and it would actually be shown by a crumpling of strata.

If ρ be the density of a body, and ϵ its modulus of linear expansion for temperature, then it is obvious that when the temperature is raised θ degrees the density becomes $\rho(1-3\epsilon\theta)$. Now suppose that a spherical shell of radius r expands so that its radius becomes $r(1+\alpha)$, and suppose that at the same time its temperature is raised θ degrees. Then, if k be the modulus of stretching, so that unit length is stretched by a length k, we have, in consequence of the above explanation of stretching,

$$\alpha = k + \epsilon\theta. \quad \ldots \ldots \quad (1)$$

Now let us consider the geometry of changes in a sphere such that a shell of internal radius r, thickness δr, and density ρ, expands until its internal radius r becomes $r(1+\alpha)$ and its density ρ becomes $\rho(1-3\epsilon\theta)$.

The external radius $r+\delta r$ clearly then becomes

$$r(1+\alpha)+\delta r\left[1+\frac{d}{dr}(r\alpha)\right].$$

Thus the mass of the shell $4\pi r^2\rho\,\delta r$ becomes

$$4\pi r^2\rho\,\delta r\left[1+2\alpha+\frac{d}{dr}(r\alpha)-3\epsilon\theta\right].$$

Then, since the mass remains unchanged, we must have

$$2\alpha+\frac{d}{dr}(r\alpha)-3\epsilon\theta=0.$$

This "equation of continuity" may clearly be written

$$\frac{d}{dr}(r^3\alpha)=3\epsilon\theta r^2.$$

* *Cf.* foot-note, p. 234.

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Substituting for α in terms of the modulus of stretching as given by (1), we have

$$\frac{d}{dr}(kr^3 + \epsilon\theta r^3) = 3\epsilon\theta r^2 .$$

Hence

$$\frac{d}{dr}(kr^3) = -\epsilon r^3 \frac{d\theta}{dr},$$

and therefore

$$k = -\frac{\epsilon}{r^3}\int r^3 \frac{d\theta}{dr}\,dr, \quad \ldots \ldots \quad (2)$$

where the integral is taken from r down to such a depth that there is no change of temperature.

If, now, θ represents the rise of temperature per unit time, and if we replace k by dK/dt, the rate of stretching per unit time, and if we make application of (2) to the case of the Earth, and write v for the temperature of the Earth at a depth x below the surface, and c for the Earth's radius, we have

$$\frac{dv}{dt} = \theta,$$

$$c - x = r;$$

and (2) becomes

$$\frac{dK}{dt} = -\frac{\epsilon}{(c-x)^3}\int_c^x (c-x)^3 \frac{d^2v}{dx\,dt}\,dx. \quad \ldots \ldots \quad (3)$$

In inserting the limits to the integral, it is assumed that the temperature at the Earth's centre is sensibly constant.

The amount by which a great circle of radius $(c - x)$ is being stretched per unit time is

$$2\pi(c-x)\frac{dK}{dt}.$$

This expression, with the above value (3) for dK/dt, is Mr. DAVISON'S result.

We know from THOMSON'S solution for the cooling of the Earth that, when x exceeds a small fraction of the Earth's radius the temperature gradient and its rate of variation in time are very small. Hence, when x is not very small, $d^2v/dx\,dt$ is very small; therefore we may with sufficient approximation replace $(c - x)^3$ under the integral sign by $c^3 - 3c^2x$. Outside of the integral we may simply neglect x. Also the limit c of the integral may be replaced by infinity, and this is desirable because THOMSON'S solution is really applicable to an infinite slab and not to a sphere. With these approximations we get

$$\frac{dK}{dt} = \epsilon\int_x^\infty \left(1 - \frac{3x}{c}\right)\frac{d^2v}{dx\,dt}\,dx. \quad \ldots \ldots \quad (4)$$

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Integrating, with regard to time, from the time t to zero, we shall get the total amount of stretching in the layer x from the epoch of consolidation down to time t. Thus

$$K = \epsilon \int_x^\infty \left(1 - \frac{3x}{c}\right) \frac{dv}{dx} dx. \qquad (5)$$

Now with THOMSON'S notation

$$\int_x^\infty \frac{dv}{dx} dx = \text{V} - v.$$

Since

$$\frac{dv}{dx} = \frac{\text{V}}{(\pi\kappa t)^{\frac{1}{2}}} e^{-x^2/4\kappa t},$$

$$\int_x^\infty x \frac{dv}{dx} dx = \frac{\text{V}}{(\pi\kappa t)^{\frac{1}{2}}} \int_x^\infty x e^{-x^2/4\kappa t} dx = \frac{2\text{V}\kappa t}{(\pi\kappa t)^{\frac{1}{2}}} e^{-x^2/4\kappa t} = 2\kappa t \frac{dv}{dx}\cdot$$

Hence (5) becomes

$$K = \epsilon \left[\text{V} - v - \frac{6\kappa t}{c^2} \cdot c \frac{dv}{dx}\right].* \qquad (6)$$

This expression gives us the total amount of contraction since consolidation in terms of the temperature and the temperature gradient. I shall return to this expression later.

Differentiating (6) with respect to the time, we have

$$\frac{dK}{dt} = \epsilon \left[-\frac{dv}{dt} - \frac{6\kappa}{c} \frac{dv}{dx} - \frac{6\kappa t}{c} \frac{d^2 v}{dx\, dt}\right].$$

But

$$\frac{dv}{dt} = -\frac{x}{2t} \frac{dv}{dx}, \qquad \text{and} \qquad \frac{d^2 v}{dx\, dt} = \frac{1}{2t}\left(\frac{x^2}{2\kappa t} - 1\right) \frac{dv}{dx}\cdot$$

Hence

$$\frac{dK}{dt} = \frac{\epsilon}{2t} c \frac{dv}{dx} \left[\frac{x}{c} - \frac{12\kappa t}{c^2} - \frac{6\kappa t}{c^2}\left(\frac{x^2}{2\kappa t} - 1\right)\right],$$

or

$$\frac{dK}{dt} = \frac{\epsilon}{2t} c \frac{dv}{dx} \left[\frac{x}{c} - \frac{3x^2}{c^2} - \frac{6\kappa t}{c^2}\right]. \qquad (7)$$

When x exceeds more than a small fraction of the Earth's radius dv/dx is very small; hence in (7) we may regard x/c as small, and write the equation

$$\frac{dK}{dt} = \frac{\epsilon}{2t} c \frac{dv}{dx} \left[\frac{x}{c} - \frac{6\kappa t}{c^2}\right]. \qquad (8)$$

* If x had not been treated as small, we should have got

$$\epsilon(\text{V} - v) \frac{c(c^2 + 6\kappa t)}{(c - x)^3} - \frac{2\kappa t}{(c - x)^3}\left[\frac{c^3 - (c - x)^3}{x} + 4\kappa t\right] \epsilon \frac{dv}{dx},$$

and it is easy to see that this leads to the same result as the above for all practical purposes.

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When $x = 0$, dK/dt is negative, and hence at the surface there is contraction or crumpling. The crumpling continues for some depth downwards, and vanishes when

$$\frac{x}{c} = \frac{6\kappa t}{c^2}.$$

Taking, with THOMSON, the foot and year as units, κ appears to be about 400; if, therefore, t is τ million years, $\kappa t = 4 \times 10^8 \tau$. Now, c being 21×10^6 feet, $\kappa\tau/c^2 = \tau/10^6$ approximately, and

$$x = \frac{24 \times 10^8}{21 \times 10^6} \tau \text{ feet},$$

$$= 114\,\tau \text{ feet}.$$

If τ be 100, $x = 2$ miles.

Thus, if the time since consolidation be 100 million years, the present depth of the stratum of no strain is 2 miles, and the depth is proportional to the time since consolidation. With a greater value of κ the depth is greater.

With regard to the value of dK/dt at greater depths, we observe that at a few miles below the stratum of zero strain $6\kappa t/c^2$ becomes negligible compared with x/c. Hence dK/dt is approximately proportional to $x\,dv/dx$. Now, if we take the figure drawn in THOMSON and TAIT'S 'Natural Philosophy,' Appendix D, and augment or diminish each ordinate NP′ proportionately to the corresponding abscissa, it is clear that we shall get just such a curve as that drawn by Mr. DAVISON. His curve might thus have been drawn without any computations at all. The function $x\,dv/dx$ is proportional to dv/dt, and thus his dotted curve is of just the same kind as the other, excepting close to the surface.

Now let us return to the expression for the integral stretching, viz.:—

$$K = \epsilon\left[(V - v) - \frac{6\kappa t}{c^2}\,c\,\frac{dv}{dx}\right] \quad \ldots \ldots \ldots \ldots (9)$$

We have

$$\frac{dv}{dx} = \frac{V}{(\pi\kappa t)^{\frac{1}{2}}}\,e^{-x^2/4\kappa t}.$$

Hence, if x be small and t large (both of which conditions apply to the present time near the Earth's surface),

$$\frac{dv}{dx} = \frac{V}{(\pi\kappa t)^{\frac{1}{2}}}.$$

Hence

$$K = \epsilon\left[V - v - \frac{6(\kappa t)^{\frac{1}{2}}}{c\pi^{\frac{1}{2}}}\,V\right] \quad \ldots \ldots \ldots \ldots (10)$$

Now, as we have seen above, with such values as those with which we have to deal, $6\kappa t/c^2$ is a small fraction, notwithstanding that it increases with the time.

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Hence, for the upper layers, we have approximately

$$K = \epsilon (\mathrm{V} - v). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Thus it appears that the integral effect is always a stretching, and that it is the same in amount at whatever speed the globe cools. The fact that, if the globe cools suddenly, the integral effect must be stretching has been pointed out by Mr. DAVISON.

If we differentiate (10), we have

$$\frac{dK}{dt} = \epsilon \left[-\frac{dv}{dt} - \frac{3\kappa}{c} \cdot \frac{\mathrm{V}}{(\pi\kappa t)^{\frac{1}{2}}} \right].$$

But

$$-\frac{dv}{dt} = \frac{x}{2t} \frac{dv}{dx},$$

and, near the surface,

$$\frac{dv}{dx} = \frac{\mathrm{V}}{(\pi\kappa t)^{\frac{1}{2}}}.$$

Hence

$$\frac{dK}{dt} = \epsilon \frac{dv}{dx}\left(\frac{x}{2t} - \frac{3\kappa}{c}\right)$$

$$= \frac{\epsilon}{2t} c \frac{dv}{dx}\left(\frac{x}{c} - \frac{6\kappa t}{c^2}\right),$$

and thus we find equation (8) again, as ought to be the case. This may also be written—

$$\frac{dK}{dt} = \frac{\epsilon}{2t} \frac{\mathrm{V}}{(\pi\kappa t)^{\frac{1}{2}}} \left(x - \frac{6\kappa t}{c}\right).$$

We must now see whether the amount of crumpling of the surface strata can be such as to explain the contortions of geological strata.

It must be borne in mind that, from a geological point of view, contraction is not the negative of stretching. When a stratum is stretched it may perhaps be ruptured, and rock may be squeezed up into the crack, at least for the strata which are very near the surface, and therefore not under great pressure; but when compressed the stratum is no doubt crumpled. Hence it is insufficient to know that the integral effect from the time of consolidation is a stretching; for that stretching may be merely the excess of a stretching over a crumpling. Now we have found above that the depth of the stratum of no strain is given by

$$x = \frac{6\kappa t}{c}.$$

Hence, at the time t', given by $t' = cx/6\kappa$, the surface of no strain was at depth x; and at all later times than t' the surface of no strain lies deeper. Therefore, to find

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the total amount of crumpling at any depth, we require to find the integral effect taken from t' to t, which is greater than t'.

The integral stretching from consolidation to time t is

$$K = \epsilon\left[V - v - \frac{6\kappa t}{c}\frac{dv}{dx}\right].$$

Now near the surface v is nearly equal to $v_0 + x\frac{dv}{dx}$;

hence

$$K = \epsilon(V - v_0) - \epsilon\frac{dv}{dx}\left[x + \frac{6\kappa t}{c}\right].$$

But

$$\frac{dv}{dx} = \frac{V}{(\pi\kappa t)^{\frac{1}{2}}}e^{-x^2/4\kappa t} = \frac{V}{(\pi\kappa t)^{\frac{1}{2}}} \text{ near the surface.}$$

Hence

$$K = \epsilon(V - v_0) - \frac{\epsilon V}{(\pi\kappa)^{\frac{1}{2}}}\left[\frac{x}{t^{\frac{1}{2}}} + \frac{6\kappa t^{\frac{1}{2}}}{c}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

At the time t', given by $t' = cx/6\kappa$,

$$K = \epsilon(V - v_0) - \frac{\epsilon V}{(\pi\kappa)^{\frac{1}{2}}}\left[x^{\frac{1}{2}}\left(\frac{6\kappa}{c}\right)^{\frac{1}{2}} + x^{\frac{1}{2}}\left(\frac{6\kappa}{c}\right)^{\frac{1}{2}}\right] = \epsilon(V - v_0) - \frac{\epsilon V}{(\pi\kappa)^{\frac{1}{2}}}\cdot 2\left(\frac{6\kappa}{c}\right)^{\frac{1}{2}}x^{\frac{1}{2}}. \quad . \quad (13)$$

This gives the total stretching from the time of consolidation until the surface of no strain has got down to x.

If, therefore, we subtract (13) from (12), we get the total stretching between the time t' and the time t, and the result is obviously—

$$K = -\frac{\epsilon V}{(\pi\kappa t)^{\frac{1}{2}}}\left[x - 2\left(\frac{6\kappa t}{c}\right)^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{6\kappa t}{c}\right].$$

This is clearly

$$K = -\epsilon\frac{dv}{dx}\left[x^{\frac{1}{2}} - \left(\frac{6\kappa t}{c}\right)^{\frac{1}{2}}\right]^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This expression is essentially negative, and therefore the total effect from t' to t is a crumpling, as was foreseen.

This integral crumpling vanishes at the same place as does dK/dt, that is to say, when

$$x = \frac{6\kappa t}{c}.$$

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This, we have shown above, will be at a depth of 2 or 3 miles. The amount of crumpling at the surface is given by putting $x = 0$, and

$$K = -\epsilon \frac{6\kappa t}{c} \frac{dv}{dx}.$$

Now we have seen that, if τ be the number of millions of years since consolidation,

$$\frac{6\kappa t}{c} = 114\,\tau \text{ feet},$$

and

$$\frac{dv}{dx} = 0^\circ\!\cdot\!02 \text{ Fahr. per foot.}$$

Hence

$$K = -\tau\epsilon \times 2{\cdot}28.$$

The total amount by which a great circle is contracted is 25,000 K miles; and, judging by the coefficient of expansion of metals, ϵ may be about 5×10^{-5}.

Thus, with these rough data, the amount of crumpling of a great circle of radius c is

$$2\pi c K = \tau \times \left(\frac{5}{10^5}\right) \times (2{\cdot}5 \times 10^4) \times 2{\cdot}28 = 2{\cdot}85\,\tau \text{ miles.}$$

Thus, in 10,000,000 years, $28\frac{1}{2}$ miles of rock would be crumpled up.

The area of rock crumpled is $4\pi c^2 . 2K$, and with these numerical data

$$4\pi c^2 . 2K = 4c\,(2\pi c\,K).$$

Now $c = 4000$ miles, and therefore

$$4\pi c^2 . 2K = 22{,}800\,\tau \text{ square miles.}$$

Thus, in 10,000,000 years, 228,000 square miles of rock will be crumpled up and piled on the top of the subjacent rocks.

The numerical data with which we have to deal are all of them subject to wide limits of uncertainty, but the result just found, although rather small in amount, is such as to appear of the same order of magnitude as the crumpling observed geologically.

The stretching and probable fracture of the strata at some miles below the surface will have allowed the injection of the lower rocks amongst the upper ones, and the phenomena which we should expect to find according to Mr. DAVISON's theory are eminently in accordance with observation. It therefore appears to me that his view has a strong claim to acceptance.